Voltage and Current Division (Kirchhoff)

In this note, we will be investigating the concepts of voltage and current division. Voltage and current division is an application of Kirchhoff’s Laws. The theoretical portion of this note is quite long. I assume that you have had no exposure to Kirchhoff’s laws. I have presented every step along the way. This theoretical section is a quick read because every step is completely presented here. There are many figures and equations. Mostly algebra so it will go quickly. Please read it all.

Kirchhoff’s Voltage Law

Kirchhoff’s Voltage Law (KVL) states that all voltage drops around any loop in any circuit sum to zero. In mathematical form,

\[ \sum_{i=1}^{n} V_i = 0 \]  

(1)

Where the \( V_i \) in Eq. 1 are the voltages across the individual components in any circuit. As an example of how to use Kirchhoff’s Voltage Law to solve a circuit, consider the circuit shown in Fig. 1. There is only one loop which we will start at the \( V_S \) – terminal. The loop goes through the DC source, \( R_1 \), \( R_2 \) and finally \( R_3 \) where it ends up at the starting point. **IMPORTANT NOTE. The loop starts, travels and ends at the spot we picked. We could have picked any starting point and traveled in either direction. The voltages and currents you calculate will have a sign that is dependent on the direction of your loop. For example if a current ends up with a negative value it means the current really flows in the direction that is opposite to the direction you picked for your loop.** This is considered a series circuit because all of the components are in series, i.e. there is only 1 loop. There are only voltages divisions in this series circuit. There is only 1 current. There is no current division.

![Figure 1](image)

If we apply Eq. (1) to the loop depicted in Fig. (1) we obtain the equation:

\[ 0 = -V_S + V_1 + V_2 + V_3 \]  

(2)

This approach produces one equation for which there are three unknowns. We need three equations to solve for three unknowns. There is an easy way to approach this dilemma. Remember:
\[ V = IR \quad (3) \]

For each voltage \( V_1, V_2 \) and \( V_3 \) shown in Fig. 1 and Eq. 2 let us substitute I times R and define the current flowing in the main loop as \( I_1 \).

\[ 0 = -V_s + R_1 I_1 + R_2 I_1 + R_3 I_1 \quad (4) \]

Let us now re-write Eq. 4 solving for \( I_1 \).

\[ I_1 = \frac{V_s}{R_1 + R_2 + R_3} \quad (5) \]

We now have one equation with one unknown, \( I_1 \), which is easily evaluated. We see that solving a KVL circuit by defining voltages in terms of current and resistance allows you to configure the current around the loop to provides an easy solution. Once this current is known you can solve for any desired quantity. To obtain the voltages as shown in Fig. 1 we use Ohm’s Law. These voltages are given as:

\[ V_1 = R_1 I_1, \quad V_2 = R_2 I_1, \quad V_3 = R_3 I_1 \quad (6) \]

**Equivalent Resistance**

Consider the circuit shown in Fig. 1 above. There are 3 resistors in series with the same current flowing through each one. We should be able to calculate a single value of resistance which is equivalent to the 3 resistors.

\[ \begin{align*}
R_1 & \quad I_1 \\
R_2 & \\
R_3 & \\
\text{I}_1 & + \quad V_s \\
\text{R}_{eq} & \\
\end{align*} \]

**Figure 2**

Figure 2 helps define the relevant current, voltage and \( R_{eq} \). The resistors have the same current, \( I_1 \), flowing through them. In this case \( V_{eq} \) is the same as \( V_s \) shown in Fig. 1. Eq. 4 becomes 7

\[ \begin{align*}
V_s &= I_1 (R_1 + R_2 + R_3) \\
V_s &= I_1 (R_{1} + R_{2} + R_{3}) \\
V_s &= I_1 R_{eq} \quad (7, 8, 9) \\
\end{align*} \]

Where

\[ R_{eq} = R_1 + R_2 + R_3 \quad (10) \]
Equation 10 giving the equivalent resistance of resistors in series can be extended to any number of resistors by adding another \( R \) term to the sum \( R_{eq} \) for each addition resistor. We can conclude the equivalent resistance of any number of resistors in series is the sum of the resistances. With this realization we can solve for the current \( I_1 \) in Fig. 1 or 2 by inspection.

\[
I_1 = \frac{V_{eq}}{R_{eq}} \quad (11)
\]

There is still one more interesting question we can ask about the circuit in Fig. 1 or 2. Is it possible to tell what voltage will be developed across any given resistor without calculating the current first? The answer is **YES!** By combining Eq. 6 and 11 we get:

\[
V_1 = V_S \frac{R_1}{R_{eq}}, \quad V_2 = V_S \frac{R_2}{R_{eq}}, \quad V_3 = V_S \frac{R_3}{R_{eq}} \quad (12)
\]

This is a very revealing result. The voltage across any resistor \( R_X \) in a series of resistors is proportional to the resistor \( R_X \) over the sum of all of the resistors, \( R_{eq} \). The ratio of the voltage \( V_X \) to \( V_S \) is in fact the same ratio as \( R_X \) over \( R_{eq} \). This relationship is given in Eq. 13. The voltage is divided based on the resistor values alone. A series circuit of resistors divides the voltage in proportion to \( R_X \) and inversely proportional to \( R_{eq} \) as shown in Eq. 12.

\[
\frac{V_X}{V_S} = \frac{R_X}{R_{eq}} \quad (13)
\]

**Kirchhoff’s Current Law**

Kirchhoff’s Current Law (KCL) is a corollary to his voltage law. It states that all currents entering and leaving any circuit node (connection point) sum to zero. Mathematically, this is:

\[
\sum_{i=1}^{n} i_i = 0 \quad (14)
\]

To solve a circuit with Kirchhoff’s Current Law, consider the circuit shown in Fig. 3 where we have arbitrarily chosen the current \( I_S, I_1, I_2, \) and \( I_3 \) directions as shown. Remember if the calculated current value is negative the current really flows in the direction opposite the one you picked. This is called a parallel circuit. There are only currents divisions in this parallel circuit. There is only 1 voltage. There is no voltage division.
Apply Eq. 14 to the node symbolized by $V_S$. Call currents into the node positive and those out of the node as negative.

\[ 0 = I_S - I_1 - I_2 - I_3 \]  
\[ I_S = I_1 + I_2 + I_3 \]

We can then substitute for the currents using Ohm’s Law. This yields:

\[ I_S = \frac{V_S}{R_1} + \frac{V_S}{R_2} + \frac{V_S}{R_3} \]

Thus we now have one equation with one unknown, $I_S$. This allows us to easily solve the circuit and obtain any desired circuit current.

\[ I_1 = \frac{V_S}{R_1}, \quad I_2 = \frac{V_S}{R_2}, \quad I_3 = \frac{V_S}{R_3} \]

**Equivalent Resistance**

For resistors in parallel, consider the circuit shown in Fig. 2 above re-notated and presented in Fig. 4.

By Kirchhoff’s Current Law the current into the node $V_S$ must equal the current out of node $V_S$:

\[ I_S = I_1 + I_2 + I_3 \]  

Using Ohms law we rewrite the currents as $V/R$ and get:
\[ I_s = \frac{V_s}{R_1} + \frac{V_s}{R_2} + \frac{V_s}{R_3} \]  
\[ I_s = V_s \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \]  
\[ R_{eq} = \frac{V_s}{I_s} \]  
\[ R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \]  

(20) factoring out \( V_s \) we get  
(21) Ohms law states  
(22) 21 and 22 yields:  
(23)

Eq. 23 for the equivalent resistance of resistors in parallel can be extended to any number of resistors by adding another 1 over \( R \) term to the denominator for each addition resistor. If only two resistors are in parallel, this formula can be reduced to a much more convenient form. Simplifying, the equivalent resistance of two resistors in parallel is given by:

\[ R_{eq} = \frac{R_1R_2}{R_1 + R_2} \]  

(24)

It isn’t a problem determining the voltage across any of the resistors in the circuit shown in Fig. 4. There is only 1 voltage, \( V_s \). What about the individual currents? That’s not a problem either.

\[ I_1 = \frac{V_s}{R_1}, \quad I_2 = \frac{V_s}{R_2}, \quad I_3 = \frac{V_s}{R_3} \]  

(25)

One last question. Can we tell what the currents \( I_1, I_2 \) and \( I_3 \) will be while only knowing the current \( I_{eq} \)? Notice Eq. 25 defines \( I_1 \) in terms of \( V_s \) and \( R_1 \). Can we find an equation with \( V_s \) in terms of a circuit current? YES! Eq. 22 can be rewritten as:

\[ V_s = I_{eq}R_{eq} \]  
\[ I_1 = I_{eq} \frac{R_{eq}}{R_1} \]  

(26) Combine with 25  
(27)

This is another revealing result. The current flowing through any single resistor, \( R_X \), in a parallel set of resistors is proportional to the sum of all of the resistors, \( R_{eq} \) over resistor \( R_X \). The ratio of the currents \( I_X \) to \( I_s \) is in fact the same ratio as \( R_{eq} \) over \( R_X \). This relationship is given in Eq. 28. The current is divided based on the resistor values alone. A parallel circuit of resistors divides the current in proportion to \( R_{eq} \) and inversely proportional to \( R_X \) as shown in Eq. 28.

\[ \frac{I_X}{I_{eq}} = \frac{R_{eq}}{R_X} \]  

(28) Parallel \( R_{eq} \)
Remember,
\[
\frac{V_X}{V_s} = \frac{R_X}{R_{eq}} \quad (13) \quad \text{Series } R_{eq}
\]

Notice that current division in a parallel circuit is the reciprocal of voltage division in a series circuit. Please note that \( R_{eq} \) in Eq. 13 is not the same as \( R_{eq} \) in Eq. 28.

That takes care of the easy circuits. Let’s look a circuit that contains both series and parallel elements. The Kirkoff laws and equivalent resistances can be utilized to solve this circuit for voltages and currents. Fig. 5 shows 1 example circuit where both laws can be applied. The derivations are not given here, but the interested student will… ;-

Just kidding. First KVL.

\[
\begin{align*}
0 &= -V_s + V_1 + V_2 \\
0 &= -V_2 + V_3 \quad (29) \\
0 &= -V_s + V_1 + V_3 \\
\end{align*}
\]

Three equations, 3 unknowns. We can solve this. It’s just algebra. This is solution number 1. There is an easier way as we saw in the series circuit example above. We used Ohms law to represent voltages as currents * resistances. Let’s define current \( I_1 \) as the loop shown by \( L_1 \) and current \( I_2 \) by the loop \( L_2 \).
\[0 = -V_s + R_1 I_1 + R_2 (I_1 - I_2) \quad (32)\]
\[0 = R_2 (I_2 - I_1) + R_3 I_2 \quad (33)\]

We don’t need a third current. We have 2 equations and 2 unknowns. We can solve this. It’s just algebra. This is solution number 2. Now let’s apply KCL. The node labeled \(V_X\) in Fig. 7 is where we will sum the currents. Recall from Eq. 14 the sum must be 0.

Once again current into the node is positive and current out of the node is negative.

\[0 = I_1 - I_2 - I_3 \quad (34)\]
\[I_1 = I_2 + I_3 \quad (35)\]

Using Ohms law we get:

\[\frac{V_s - V_X}{R_1} = \frac{V_X}{R_2} + \frac{V_X}{R_3} \quad (36)\]

We have 1 equation and 1 unknown. We can solve this. It’s just algebra. This is solution number 3. This appears to be the simplest method of the 3 derived from Kirchhoff’s laws.

Now use equivalent resistances to find the circuit parameters. We can find an equivalent resistance for the parallel pair \(R_2\) and \(R_3\) using Eq. 24.

\[R_p = \frac{R_2 R_3}{R_2 + R_3} \quad (37)\]

Now we can get an equivalent resistance for the series pair \(R_1\) and \(R_p\).

\[R_s = R_1 + R_p \quad (38)\]

With \(R_s\) we get \(I_1\) from Ohms law. \(V_X\) comes from the voltage divider Eq. 12 with \(R_{eq}\) given by Eq. 38. With \(V_X\), the currents \(I_2\) and \(I_3\) come from Eq. 36. Here is a 4th method to determine the circuit parameters. The one you use is really your choice. Some are less work and presumably have less chance for error. End of diatribe!